Abstract - Left almost semigroups (LA-semigroups) or Abel-Grassmann's groupoids (AG-groupoids) have been studied by several authors and this motivated to extend these concepts to Left Almost ring (LA-ring), which carries attraction due to its structural formation. In this paper we generalize the structure of commutative semigroup ring (ring of semigroup $S$ over ring $R$ represented as $R[X;S]$) to a non-associative LA-ring of LA-semigroup $S$ over LA-ring $R$ represented as $R[X';S \in S]$, consisting of finitely nonzero functions. Nevertheless it also possesses associative ring structures. Furthermore we also discuss the LA-ring homomorphisms.

KeyWords - LA-Ring, LA-Group, LA-Semigroup And Commutative Semigroup Ring. 2000 Msc: 20m10, 20n99, 16y99.

INTRODUCTION

Following M. Kazim and M. Naseerudin [3], a groupoid $(S,\ast)$ is called a left almost-semigroup (LA-Semigroup) or Abel-Grassmann's groupoid (AG-groupoid) (see [2]), if it satisfies the left invertive law; $(a \ast b) \ast c = (c \ast b) \ast a$ for all $a, b, c \in S$. By [3], every LA-semigroup $(S,\ast)$ is medial that is $(a \ast b) \ast (c \ast d) = (a \ast c) \ast (b \ast d)$ for all $a, b, c, d \in S$. Whereas an LA-semigroup $(S,\ast)$ is said to be weak associative LA-semigroup if $(a \ast b) \ast c = b \ast (a \ast c)$ or $(a \ast b) \ast c = b \ast (c \ast a)$ for all $a, b, c \in S$. Q. Mushtaq and M. S. Kamran extended the notion of LA-semigroup to left almost group (LA-group) (see [6]). A groupoid $(G,\ast)$ is called a left almost group (LA-group), if there exists left identity $e \in G$ such that $e \ast a = a$ for all $a \in G$, for $a \in G$ there exists $b \in G$ such that $b \ast a = e$ and invertive law holds in $G$.

In [10], T. Shah and I. Rehman have discussed LA-rings of finitely nonzero functions. By a left almost ring (LA-ring), we mean a nonempty set $R$ with at least two elements such that $(R,\ast)$ is an LA-group and $(R,\cdot)$ is an LA-semigroup and both left and right distributive laws hold. For example, from a commutative ring $(R,\ast,\cdot)$, we can always obtain an LA-ring $(R,\oplus,\cdot)$ by defining for $a, b, c \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. Furthermore in [10], T. Shah and I. Rehman have introduced the notion of LA-module, a non abelian non associative structure, over an LA-ring. In continuation, T. Shah and M. Raees [9], have generalized some useful results analogous to module theory for LA-modules. The study [10], addresses the structure of left almost rings (LA-rings) of finitely nonzero functions from commutative semigroup $S$ to LA-ring $R$, which in fact generalizes various analogous of corresponding parts of commutative semigroup rings. In the continuation of [10], in this study commutative semigroup $S$ is taken as an LA-semigroup and almost all the established results in [10] stand as a particular case.

THE CONSTRUCTION

In this section we construct an LA-ring of finitely non-zero functions from LA-semigroup to LA-ring and also we discuss the concepts of degree and order of LA-ring. Let $(R,\ast,\cdot)$ be an LA-ring with left identity and $S$ be an LA-semigroup under $\ast$. We name the set \{\(f : f : S \to R\), where $f$ are finitely nonzero\} as $T$.

Define the binary operation $+ \in T$ as $(f + g)(s) = f(s) + g(s)$. $(T,+)$ is LA-group. Indeed, let $f, g \in T$. As $f(s), g(s) \in R$ for all $s \in S$. So $(f + g)(s) = f(s) + g(s) \in R$ and hence $f + g \in T$.

Let $f, g, h \in T$. As $f(s), g(s), h(s) \in R$ and left invertive law holds in $(R,\ast)$. So we have

\[
\begin{align*}
&f \ast g \ast h = f \ast (g \ast h) = f \ast g \ast h,
&f \ast (g \ast h) = (f \ast g) \ast h.
\end{align*}
\]

hence $g \ast h \in T$. 
Thus left invertive law holds in $T$. Define the map $o: S \to R$ such that $o(s) = 0$ for all $s \in S$.

This implies $o \not\subseteq f$. Thus $o$ is left additive identity in $T$. For every $f \in T$ there exists a function $-f: S \to R$ defined by for all $s \in S$

$((-f) + f)(s) = (-f(s)) + f(s) = -f(s) + f(s) = 0$.

This implies $o/(f \not\subseteq o)$. So the left inverses exist in $T$. Hence $(T, +)$ is an LA-group. We can say $f + (-f) = 0$ as $-f(s)$ is also the right inverse of $f(s)$ in $R$ by [6]. Now we define binary operation $\ell$ in $T$ as follows

$$f \ell g \equiv f \circ g \equiv f + (-g)\quad r(u)$$

We claim that $T, \ell$ is an LA-semigroup. As for $f \in T$ and $g \in T$, where $t \in T, u \not\subseteq T, \ell$ and $\ell, 0$ is LA-ring, $f \ell g \subseteq T$. Since $f, g$ are finitely nonzero on $S$, therefore $f \ell g \subseteq T$.

Let $f, g, h \in T$ and $s \in S$ such that

$$f \ell (g \ell h) = (f \ell g) \ell h \quad r(u)$$

Hence $T, \ell, 0 \ell \ell$.

Thus left invertive law holds in $T$. Now we have to verify that the binary operation $\ell$ is distributive over addition. As $f \ell (g \ell h)$ and $h(u) \in R$ and multiplication is distributive over addition in $R$, so

$$f \ell (g \ell h) = (f \ell g) \ell h$$

Similarly

Thus $(T, \ell)$ is an LA-ring of LA-semigroup $(S, \not\subseteq 0)$ over LA-ring $(R, +, \cdot)$.

Degree and order of elements of LA-ring $R \ell S, s \not\subseteq S$

The concept of degree and order are not generally defined in semigroup rings unless we have to consider $S$, a totally ordered semigroup with $0$ adjoined that is ordered monoid). The structure of LA-ring $R \ell S, s \not\subseteq S$ is also not convenient for defining degree and order of an element unless $S$ is totally ordered. The concept of total ordering of LA-semigroups has been discussed by T. Shah, I. Rehman and A. Ali (see [8]).

Here we define support of $f = \sum_{i=1}^{n} f_i X^{s_i}$ abbreviated as $\text{Supp}(f) = \{s_i : f_i \neq 0\}$. The order and degree of $f$ is defined as $\text{ord}(f) = \min \text{Supp}(f)$ and $\text{deg}(f) = \max \text{Supp}(f)$.

Let $(\mathbb{Q}^+, \cdot)$ denote the group of all positive rational numbers. If we take $S = (\mathbb{Q}^+)^{\text{ILAS}}$, where ILAS abbreviates Initial LA-semigroup, which is made an LA-semigroup by defining the binary operation $\star$

as

$$a \star\triangleleft b = 0 \text{ if } a \not\subseteq 0 \text{ or } b \not\subseteq 0,$$

$$b \triangleleft a^{-1} \text{ if } a \not\subseteq 0 \text{ and } b \not\subseteq 0.$$
In a polynomial ring $R[X]$, for $f, g \in R[X]$, 
\[ \deg(f \circ g) = \deg(f) \cdot \deg(g) \text{ and} \]
\[ \ord(f \circ g) = \ord(f) \cdot \ord(g) \]

If $R$ is an integral domain, then 
\[ \deg(f \circ g) = \deg(f) \cdot \deg(g) \] and 
\[ \ord(f \circ g) = \ord(f) \cdot \ord(g) \]

But the following lemma shows a deviation for LA-ring $R[X; s \in S]$ of LA-semigroup $(Q^+_0, *) = S$ over LA-ring $R$.

### MAIN RESULTS

In this section we generalize some results in [1]. Specifically, we show the necessary and sufficient condition for an LA-ring $R[X; s \in S]$ to be an LA-integral domain. We also discuss the homomorphisms of LA-rings.

Following external direct sum of semigroups as in [1], we define external direct sum of LA-semigroups $(S, \circ)$ and $(T, \#)$ as 
\[ S \oplus T \ni (0, t) \ni (s, t) \ni S \times T \]

whereas the binary operation in $S \oplus T$ is defined as 
\[ (0, t_1) \cdot (0, t_2) = (0, t_1 + t_2) \text{ for } s_1, s_2 \in S \text{ and } t_1, t_2 \in T. \]

A mapping $\mathcal{E}$ of an LA-ring $(R, +, \cdot)$ into an LA-ring $(R', +, \cdot)$ is called a homomorphism if 
\[ \phi(a + b) = \phi(a) + \phi(b) \]
and 
\[ \phi(a \cdot b) = \phi(a) \cdot \phi(b) \]
and for the ideal $I$ of $R$, the mapping $\nu : R \rightarrow R/I$ defined as 
\[ \nu(a) = a + I \]
is the natural epimorphism of LA-ring $\nu(R)$ onto $R/I$. Let $\theta$ be an epimorphism of an LA-ring $R$ to an LA-ring $R'$, then $\nu(R)/\ker(\theta) \cong R'$.

The following is a generalization of [10].

**Theorem:** Let $R$ be an LA-ring and $S$, $T$ be LA-semigroups. Then 
\[ R \oplus S \oplus T \ni (0, 0, 0) \ni (s, t, u) \ni S \times T \]

where $S \oplus T$ the is external direct sum of LA-Semigroups $S$ and $T$.

**Theorem:** The polynomial LA-ring $R[\{Y_{\lambda}\}_{\lambda \in \Lambda}]$, 
where $R$ is an LA-ring and $\{Y_{\lambda}\}_{\lambda \in \Lambda}$ is a family of commuting indeterminates and $F = \sum_{\lambda \in \Lambda} Z_{\lambda}$ such that $Z_{\lambda} \neq Z_0$. Then $R \lhd \mathbb{F}_{\lambda}$ is isomorphic to LA-ring $R[X; F]$ of free commutative semigroup $(F, +)$ over $R$.

By [7], a semigroup $S$ is said to be $m$-torsion free if for any $x, y \in S$ there exists $m \geq 1$ with $x^m = y^m$, then $x = y$. We extend this for $\mathbb{L}^3$-semigroup with left identity $e$.

**Definition:** An $\mathbb{L}^3$-semigroup $(S, \circ)$ with left identity $e$ is said to be an $M$-torsion free if for all $x, y \in S$ there exists a subset $M$ of $\mathbb{Z}^+$ such that $1 \leq m \in M$ with $x^m = y^m$ implies $x = y$.

**Example:** $(Q^+_0, \circ)$ is an $\mathbb{L}^3$-semigroup with left identity 1, defined as 
\[ a \circ b \equiv 0 \text{ if } a \equiv 0 \text{ or } b \equiv 0, \]
\[ a \equiv b \equiv a \equiv 1 \text{ if } a \equiv 1 \text{ and } b \equiv 0, \]
is an $O$-torsion free where $O$ is the set of odd positive integers. For this, consider $m = 3$. Let $x^3 = y^3$. As $(Q^+_0, \circ)$ is an $\mathbb{L}^3$-semigroup, so $x^2 \circ x = y^2 \circ y$. This implies $x \equiv y$. Hence $Q^+_0 \equiv O$ is an $O$-torsion free $\mathbb{L}^3$-semigroup.

Similarly $\mathbb{Z}$, $\mathbb{Q}$ an $\mathbb{L}^3$-semigroup with left identity 0 defined as $a \circ b = b - a$, is an $O$-torsion free where $O$ is the set of odd positive integers.

The following is a generalization of [10].

**Theorem:** Let $R$ be an LA-ring with left identity and let $(S, \circ)$ be an $\mathbb{L}^3$-semigroup. Let 
\[ R[X; s \in S] \]
be an LA-ring of $S$ over $R$. Then 
\[ R[X; s \in S] \]
is an LA-integral domain if and only if $R$ is an LA-integral domain and $S$ is an $M$-torsion free and cancellative.

**Theorem:** Let $\mu : R \rightarrow R_0$ be an LA-ring homomorphism. Let $A = \ker \mu$ and $\phi : S \rightarrow S_0$ be an LA-semigroup homomorphism, where $S, S_0$
are LA-semigroups with $0$ adjoined to them. Then the following statements holds:

1. $\mu^* : R[X^s; s \in S] \to R_0[X^s; s \in S]$ is defined as $\mu^*(\sum_{i=1}^{n} r_i X^{s_i}) = \sum_{i=1}^{n} \mu(r_i) X^{s_i}$, is LA-ring homomorphism such that $\ker \mu^* = A[X^s; s \in S] = \ker \mu[X^s; s \in S]$.

$\mu^*$ is surjective if $\mu$ is surjective.

2. $\varphi^* : R[X^s; s \in S] \to R[X^s; s \in S_0]$ defined as $\varphi^*(\sum_{i=1}^{n} r_i X^{s_i}) = \sum_{i=1}^{n} r_i X^{\varphi(s_i)}$, is surjective and $\ker \varphi^* = I$. The ideal of $R[X^s; s \in S]$ generated by $\langle X^a \preceq r X^b : (a, b) \in \mathcal{O}, X \rangle \in R \rangle$ is surjective if $\varphi$ is surjective.

$\langle X^a \preceq r X^b : (a, b) \in \mathcal{O}, X \rangle \in R \rangle$ is an LA-ring homomorphism such that $\ker \tau = \ker \mu^* + \ker \varphi^* = A[X^s; s \in S] + I$. Then $\tau$ is surjective if $\mu$ and $\varphi$ are surjective.

**Definition:** An ideal $P$ of an LA-ring $R$ is called prime if and only if $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$, where $A$ and $B$ are ideals in $R$.

The following is a generalized form of [10].

**Corollary:** Let $A$ be a proper ideal of LA-ring $R$, then $A[X^s; s \in S]$ is prime ideal in $R[X^s; s \in S]$ if and only if $A$ is prime ideal in $R$ and $S$ is cancellative $M$-torsion free LA-semigroup.

**REFERENCES**


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