CHARACTERISTICS OF PERMUTATION GRAPHS USING AMBIVALENT AND NON-AMBIVALENT CONJUGACY CLASSES

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Abstract -The aim of this paper is to study special graphs of permutation graphs. In this paper, we introduce two kind of permutation graphs $G_{\hat{c}}$ and G_{ψ}. The first on $G_{\hat{c}}$ based on the number of disjoint cycle factors without the 1-cycle of permutation is given and the second one G_{ψ} based on ambivalent and non-ambivalent conjugacy classes in alternating groups. Then we study some of the essential properties of permutation graphs $G_{\hat{c}}$ and G_{ψ} with their permutations as the vertices and discuss their structures as conjugacy classes in symmetric and alternating groups. In particular, we study the connected permutation graphs and connected permutation components of graph theory. Also, several examples are given to illustrate the concepts introduced in this paper.

Keywords: Graph theory, alternating groups, conjugacy classes, permutations, ambivalent groups.
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I. INTRODUCTION

A graph $G=(V,E)$ is connected if there is a path from any point to any other point in the graph [3]. A graph that is not connected is said to be disconnected. For two vertices v_1 and v_2 ($v_1 \neq v_2$), the distance between v_1 and v_2 is the number of edges in a shortest path joining v_1 and v_2. The diameter of graph G is the maximum distance between any two vertices of G. An edge between just one vertex is called a loop [5]. The study of permutations and combinations is at the root of several topics in mathematics such as graph theory, number theory, algebra, and many other specialties.Any pair of permutations λ and β in S_n are conjugate iff they have the same cycle type [2]. The conjugacy class of $\beta \in S_n$ is referred by $C^{\alpha}(\beta)$, where C^{α} depends on the cycle partition α of its elements. If this class splits into two conjugacy classes of A_n, we denote these by $C^{\alpha\pm}$. So A_n is ambivalent group iff each of $C^{\alpha\pm}$ in A_n are ambivalent.If $\beta=(k_1,k_2,...,k_r)$, then $|\beta|=r$ and we obtain $\alpha(\beta)=\alpha(\beta^{-1})$. Since $(k_1,k_2,...,k_r)^{-1}=(k_r,...,k_2,k_1)$, every permutation in S_n is conjugate to its inverse, and the groups in which each element is a conjugate of its inverse are called ambivalent, we can see that S_n is an ambivalent group,but that is not true in general for alternating group A_n, where if we assume $\theta=\{1,2,5,6,10,14\}$ we have (A_n, $n\in\theta$) are ambivalent groups and (A_n, $n\notin\theta$) are non-ambivalent ones [6].Suppose, first, that $\beta \in S_n-\{1\}$. Then $\text{supp}(\beta)$, the support of β, is the set $\{i\in\Omega \mid \beta(i)\neq 1\}$ where $\Omega=\{1,2,...,n\}$. So we say β and λ are disjoint cycles iff $\text{supp}(\beta)\bigcap\text{supp}(\lambda)=\phi$ [6]. In recently years, the permutations have been researched in many fields like algebras, topological spaces and others see ([9]-[16]). In this work we applied permutations in graph theory, we studied special graphs of permutation graphs and introduced two kind of permutation graphs $G_{\hat{c}}$ and G_{ψ}. The first on $G_{\hat{c}}$ based on the number of disjoint cycle factors without the 1-cycle of permutation is given and the second one G_{ψ} based on ambivalent and non-ambivalent conjugacy classes in alternating groups. Then we study some of the essential properties of permutation graphs $G_{\hat{c}}$ and G_{ψ} with their permutations as the vertices and discuss their structures as conjugacy classes in symmetric and alternating groups. In particular, we study the connected permutation graphs of graph theory. Also, several examples are given to illustrate the concepts introduced in this work.

II. PRELIMINARIES

We now begin by recalling some definitionswhich are needed in our work.

Definition 2.1: [17, 18]
Each $\lambda \in S_n$ can be written as $\gamma_1\gamma_2....\gamma_{c(\lambda)}$. With γ_i disjoint cycles of length α_i and $c(\lambda)$ is the number of disjoint cycle factors including the 1-cycle of λ. A partition $\alpha = \alpha(\lambda) = (\alpha_1(\lambda), \alpha_2(\lambda), ..., \alpha_{c(\lambda)}(\lambda))$ is called cycle type of λ.

Definition 2.2: [2]
Let α be a partition of n. We define $C^\alpha \subset S_n$ to be the set of all elements with cycle type α. Also, we denoted to C^α by $C^\alpha(\beta)$, if $\beta \in C^\alpha$.

Theorem 2.3: [4]
Let $\beta \in C^\alpha$ in S_n ($n > 1$). Then $C^\alpha(\beta)$ does not splitiff either the nonzero parts of $\alpha(\beta)$ are not all different or some of them are even. ($C^\alpha(\beta)$ splits into two A_n-classes $C^{\alpha\pm}$ of equal orderin any other case).

Definition 2.4: [20, 21]A graph G is a pair (V, E), where V is a finite set and E is a collection of unordered pairs in $V \times V$. The elements of V are referred to as the vertices of the graph and the elements of E are referred to as the edges.

Definition 2.5: [7, 8] A graph $G = (V, E)$ is connected if any of vertices x, y of G are connected by a path in G; otherwise, the graph is disconnected.

Some of Results on Permutations 2.6(see [6])

(1) $(k_1, k_2, ..., k_r) \in A_n \Leftrightarrow r$ is odd.

(2) $\beta \in A_n \Leftrightarrow n - c(\beta)$ is even, so $\text{sgn}(\beta) = (-1)^{n-c(\beta)}$.

(3) $|\beta| = l.c.m\{\alpha_i(\beta) : 1 \le i \le c(\beta)\}$, the order of a cycle β is equal to the least common multiple of the set $\{\alpha_i(\beta) : 1 \le i \le c(\beta)\}$.

(4) If μ and λ are even permutations, then $\mu\lambda \in A_n$. Also, μ is conjugate to λ in A_n (i.e $\mu \underset{A_n}{\approx} \lambda$), if there exists even permutation $\beta \in A_n$ such that $\beta\mu\beta^{-1} = \lambda$.

(5) A_n is ambivalent if and only if $(\lambda \underset{A_n}{\approx} \lambda^{-1})$, for any λ in A_n.

(6) $\text{supp}(\mu) \cap \text{supp}(\lambda) = \phi \Leftrightarrow \text{supp}(\mu^{-1}) \cap \text{supp}(\lambda^{-1}) = \phi \Leftrightarrow \mu$ and λ are disjoint cycles.

(7) If $\text{supp}(\mu) \cap \text{supp}(\lambda) = \phi$, then $\mu\lambda = \lambda\mu$.

Theorem 2.7: [4]
For any $\lambda \in C^+ \cup C^+$ in A_n, let $\varpi(\lambda) = \{\alpha_m \mid \alpha_m \equiv 3 \text{ (mod 4)}; \forall \alpha_m \text{ of } \alpha(\lambda)\}$. Then $C^{\alpha\pm}$ are ambivalent if and only if $|\varpi(\lambda)|$ is even.

Lemma 2.8: [1]
$A_1, A_2, A_5, A_6, A_{10}$ and A_{14} are the only ambivalent alternating groups.

III. CHARACTERISTICS OF PERMUTATION GRAPH

Definition: 3.1
Let $\pi_1\pi_2....\pi_{c(\lambda)}$ be a product of the disjoint cycles with $\pi_1\pi_2...\pi_{c(\lambda)} = \lambda \in S_n$. We say $\hat{c}(\lambda)$ is the number of disjoint cycle factors without the 1-cycle of λ.

Remark: 3.2
For any permutation λ in symmetric group S_n, we have $c(\lambda) \ge \hat{c}(\lambda)$.

Example: 3.3
Let $\beta = (1\ \ 6)(2\ \ 3)(5\ 9) \in S_{10}$, Then $c(\beta) = 7 > 3 = \hat{c}(\beta)$.

Definition: 3.4
Let $(S_n, \circ)$ be a symmetric group. Then for any two permutations $\gamma, \lambda \in S_n$, we say they are adjacent under ∂ and denoted by $(\gamma, \lambda)^{\partial_n}$, for some characterization ∂ on permutations. This term $(\gamma, \lambda)^{\partial_n}$ is hold for any $\gamma \ne \lambda \in S_n$ iff $\partial(\gamma) = \partial(\lambda)$. Also, $(\lambda, \lambda)^{\partial_n}$ is hold for any $\lambda \in S_n$ iff $\partial(\lambda^2) = \partial(\lambda)$.

Example: 3.5
Let $\gamma = (1\ \ 15)(2\ \ 12)(11\ \ 9)(13\ \ 8)$ and $\lambda = (2\ \ 11)(10\ \ 7)(5\ \ 1\ \ 6)(3\ \ 9\ \ 14)$ be two permutations in symmetric group S_{15} and let $\partial = \hat{c}$. Since $\hat{c}(\gamma) = 4 = \hat{c}(\lambda)$, then γ and λ are adjacent under $\hat{c}$ [i.e, $(\gamma, \lambda)^{\hat{c}_{15}}$].

Definition: 3.6
Let $(S_n, \circ)$ be a symmetric group and let $H_n = \{\lambda_i \mid i = 1, \ldots, k\ ; \lambda_i \ne e\}$ be a subset of S_n where e is identity permutation of symmetric group S_n. A permutation graph G_∂ is a pair (H_n, E), where E is a collection of unordered

pairs in $H_n \times H_n$. The elements of H_n are referred to as the vertices of the graph and the elements of E are referred to as the edges. Also, any two vertices λ_i, λ_j in H_n are adjacent $(\gamma,\lambda)^{\partial}{}_n$ iff $(\lambda_i,\lambda_j)\in E$. Also, (λ_i,λ_i) is called a loop under ∂ iff $(\lambda_i,\lambda_i)\in E$ [i.e, $\partial(\lambda_i{}^2)=\partial(\lambda_i)$].

Example: 3.7

Let $G_{\partial}=G_{\widehat{c}}$ and $H_{13}=\{\lambda_1,\lambda_2,\ldots,\lambda_7\}$ be a subset of S_{13} , where λ_i is defined by

$$\lambda_i=\begin{cases}(i\quad i+1),\ if\ \ i\ is\quad prime\ number,\\ \ (i\quad i+1)(13-i\quad 13),\ if\ \ W.O.\end{cases},$$

for all $1\le i\le 7$.

Then

$\widehat{c}(\lambda_1)=\widehat{c}(\lambda_4)=\widehat{c}(\lambda_6)=2,\widehat{c}(\lambda_2)=\widehat{c}(\lambda_3)=\widehat{c}(\lambda_5)=\widehat{c}(\lambda_7$ [illegible]

Also, $\widehat{c}(\lambda_i{}^2)=0,\forall\lambda_i\in H_{13}$. So, there is no loop [since $\widehat{c}(\lambda_i{}^2)\ne\widehat{c}(\lambda_i),\forall\lambda_i\in H_{13}$]. Then

$E=\{(\lambda_1,\lambda_4),(\lambda_1,\lambda_6),(\lambda_4,\lambda_6),(\lambda_2,\lambda_3),(\lambda_2,\lambda_5),$ [illegible] $,(\lambda_3,\lambda_7),(\lambda_5,\lambda_7)\}$

We represent a permutation graph G_{∂} with the picture in Figure 1.

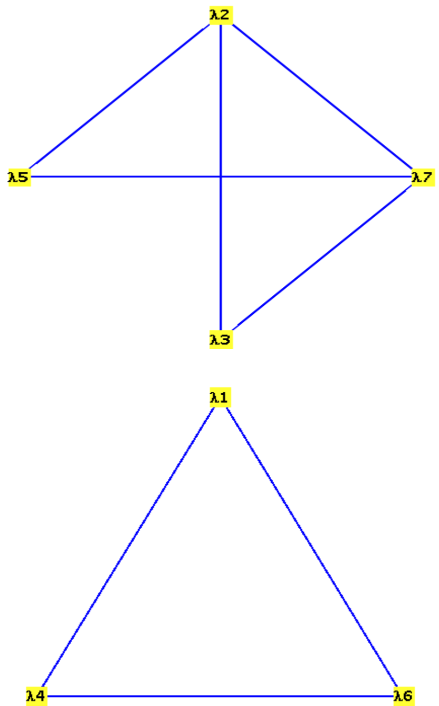

Figure 1: A graph with 7 vertices and 9 edges.

Remark: 3.8

Two edges are adjacent if they share a common vertex. A path from λ_i to λ_j in a permutation graph is a sequence of adjacent edges such that λ_i is in the first edge of the sequence and λ_j is in the last edge of the sequence.

Example: 3.9

Let $H_8=\{\lambda_1,\lambda_2,\lambda_3\}$ be a subset of S_8 , where λ_i is defined by $\lambda_i=(i\quad i+1),\forall 1\le i\le 3$. Then $\widehat{c}(\lambda_i)=2,\forall 1\le i\le 3$. Therefore, in Figure 2 the path $(\lambda_1,\lambda_2)(\lambda_2,\lambda_3)$ from λ_1 to λ_3.Also, $\widehat{c}(\lambda_i{}^2)=0\ne 2,\forall 1\le i\le 3$. So, this permutation graph has no loop.

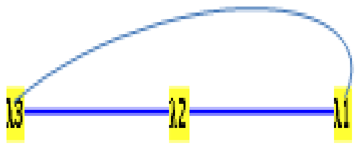

Figure 2: A path from vertex λ_1 to vertex λ_3.

Definition: 3.10

We say a permutation graph is permutation connected if for every pair of vertices there exists a path between them. All the permutation graphs we have seen so far are permutation connected. A permutation disconnected graph consists of multiple connected pieces called permutation components.

Example: 3.11

See example (3.5), we have Figure 1 shows a permutation graph that is not permutation connected with 2 connected permutation components.

Remarks: 3.12

Let $H_n=\{\lambda_1,\lambda_2,\ldots,\lambda_k\}$ be a subset of S_n and $G_{\partial}=G_{\widehat{c}}$. Then;

1. G_{∂} is a permutation connected graph if and only if $\partial(\lambda_i)=\partial(\lambda_j)$, for all $1\le i,j\le k$.
2. G_{∂} has at least two connected permutation components if and only if $\partial(\lambda_i)\ne\partial(\lambda_j)$, for some $1\le i,j\le k$.
3. G_{∂} is a permutation disconnected graph if and only if $\partial(\lambda_i)\ne\partial(\lambda_j)$, for some $1\le i,j\le k$.

Definition: 3.13

Let $\beta\in S_n$, we define image β under the map

$$\psi: S_n\to\{0,1\}\text{ by }\psi(\beta)=\begin{cases}1, & if\ \ \beta\underset{A_n}{\approx}\beta^{-1},\\ 0, & if\quad O.W.\end{cases}$$

Lemma: 3.14

Let $H_n = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ be a subset of S_n and $G_\partial = G_\psi$. Then G_∂ is a permutation connected graph if $\lambda_i \notin A_n$, for all $1 \le i \le k$.

Proof. Since $\lambda_i \notin A_n$ for all $1 \le i \le k$ and $G_\partial = G_\psi$, then $\partial(\lambda_i) = 0$ for all $1 \le i \le k$ and this implies that G_∂ is a permutation connected graph.

Lemma: 3.15

Let $H_n = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ be a subset of S_n and $G_\partial = G_\psi$. Then G_∂ is a permutation connected graph if $n \in \theta = \{1,2,5,6,10,14\}$.

Proof. Assume that $n \in \theta = \{1,2,5,6,10,14\}$ and $G_\partial = G_\psi$, then by lemma (2.8) we consider that $\partial(\lambda_i) = 1$ for all $1 \le i \le k$ and this implies that G_∂ is a permutation connected graph.

Lemma: 3.16

Let $H_n = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ be a subset of S_n and $G_\partial = G_\psi$. Then G_∂ is a permutation connected graph if for any $1 \le i \le k$, the partition $\alpha(\lambda_i) = (\alpha_1, \alpha_2, \ldots, \alpha_{c(\lambda_i)})$ such that $|\varpi(\lambda_i)|$ is even.

Proof.

Assume $G_\partial = G_\psi$ and for any $1 \le i \le k$, the partition $\alpha(\lambda_i) = (\alpha_1, \alpha_2, \ldots, \alpha_{c(\lambda_i)})$ such that $|\varpi(\lambda_i)|$ is even. Then by lemma (2.7) we consider that $\partial(\lambda_i) = 1$ for all $1 \le i \le k$ and this implies that G_∂ is a permutation connected graph.

Lemma: 3.17

Let $H_n = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ be a subset of S_n and $G_\partial = G_\psi$. Then G_∂ is a permutation disconnected graph if for some $1 \le i \ne j \le k$ such that $\lambda_i \notin A_n$ and $|\varpi(\lambda_i)|$ is even for $\alpha(\lambda_j)$.

Proof. Suppose that for some $1 \le i \ne j \le k$ such that $\lambda_i \notin A_n$ and $|\varpi(\lambda_i)|$ is evenfor $\alpha(\lambda_j)$.Since $\lambda_i \notin A_n$, for some $1 \le i \le k$ and $G_\partial = G_\psi$.Then $\partial(\lambda_i) = 0$. Also, since $|\varpi(\lambda_i)|$ is evenfor $\alpha(\lambda_j)$.Thus $\partial(\lambda_j) = 1$. Therefore, we have neither $\partial(\lambda_i) = 0$ for all $1 \le i \le k$ nor $\partial(\lambda_i) = 1$ for all $1 \le i \le k$. Hence G_∂ is a permutation disconnected graph.

Remark 3.18

By above lemma we have the permutation disconnected graph G_ψ has at least two connected permutation components.

Lemma : 3.19

Let $G_{\hat{c}} = (H_n, E)$ and $G_\psi = (H_n, E')$ be two permutation graphs. Then for any pair $\lambda_i \ne \lambda_j \in H_n$, we have:

(1) If $\lambda_i \underset{S_n}{\approx} \lambda_j$, then $\hat{c}(\lambda_i) = \hat{c}(\lambda_j)$.

(2) If $\lambda_i \underset{A_n}{\approx} \lambda_j$, then $\psi(\lambda_i) = \psi(\lambda_j)$ and $\hat{c}(\lambda_i) = \hat{c}(\lambda_j)$.

Proof: (1) Suppose that $\lambda_i \underset{S_n}{\approx} \lambda_j$, then they have the same structure in symmetric group S_n. That means they have the same partition. Therefore, we consider that $\alpha(\lambda_i) = \alpha(\lambda_j) \Rightarrow$
$(\alpha_1(\lambda_i), \alpha_2(\lambda_i), \ldots \alpha_{c(\lambda_i)}(\lambda_i)) = (\alpha_1(\lambda_j), \alpha_2(\lambda_j), \ldots \alpha_{c(\lambda_j)}(\lambda_j))$.
This implies that $c(\lambda_i) = c(\lambda_j) = m > 1$ and $\alpha_k(\lambda_i) = \alpha_k(\lambda_j)$, $\forall 1 \le k \le m$. Then $\hat{c}(\lambda_i) = \hat{c}(\lambda_j)$.

(2)Also, if $\lambda_i \underset{A_n}{\approx} \lambda_j$ …..(i), then $\lambda_i^{-1} \underset{A_n}{\approx} \lambda_j^{-1}$ ……(ii). Now, either $\psi(\lambda_i) = 1$ or $\psi(\lambda_i) = 0$, when $\psi(\lambda_i) = 1$, we have $\lambda_i \underset{A_n}{\approx} \lambda_i^{-1}$, but from (ii) we consider $\lambda_i \underset{A_n}{\approx} \lambda_j^{-1}$ ……(iii) (since $\approx$ is a transitive). From (i) ($\lambda_i \underset{A_n}{\approx} \lambda_j \Rightarrow \lambda_j \underset{A_n}{\approx} \lambda_i$) and (iii) ($\lambda_i \underset{A_n}{\approx} \lambda_j^{-1}$), we get $\lambda_j \underset{A_n}{\approx} \lambda_j^{-1}$. Then $\psi(\lambda_i) = 1 = \psi(\lambda_j)$. Furthermore, if $\psi(\lambda_i) = 0$, we consider that λ_i and λ_i^{-1} are not conjugate in alternating group A_n, let $\psi(\lambda_j) \ne 0$, then $\psi(\lambda_j) = 1$ and hence $\lambda_j \underset{A_n}{\approx} \lambda_j^{-1}$. This implies that

$\lambda_i \underset{A_n}{\approx} \lambda_i^{-1}$ (since $\lambda_i \underset{A_n}{\approx} \lambda_j$). That means $\psi(\lambda_i)=1$. But this contradiction. Therefore $\psi(\lambda_j)=0$ and hence $\psi(\lambda_i)=0=\psi(\lambda_j)$. Finally, since $\lambda_i \underset{A_n}{\approx} \lambda_j$, then they have the same structure in alternating group A_n. That means they have the same partition. Therefore, we consider that $\alpha(\lambda_i)=\alpha(\lambda_j) \Rightarrow (\alpha_1(\lambda_i),\alpha_2(\lambda_i),...,\alpha_{c(\lambda_i)}(\lambda_i)) = (\alpha_1(\lambda_j),\alpha_2(\lambda_j),...,\alpha_{c(\lambda_j)}(\lambda_j))$. This implies that $c(\lambda_i)=c(\lambda_j)=m>1$ and $\alpha_k(\lambda_i)=\alpha_k(\lambda_j)$, $\forall 1\le k\le m$. Then $\hat{c}(\lambda_i)=\hat{c}(\lambda_j)$.

Theorem 3.20:
Let $G_\psi=(H_n,E)$ be a permutationgraphand $H_n=C^{\alpha+}\bigcup C^{\alpha-}$, where $C^{\alpha\pm}$ are the conjugacy classes of A_n. Then G_ψ is a permutation connected graph if $4\,|\,(\alpha_i-1)$ for each parts α_i of α for any the partition of $\beta\in H_n$.

Proof:
In the first we need to prove that for each permutation β in $C^{\alpha-}$ or $C^{\alpha+}$ is conjugate to its inverse in A_n (i.e $\beta \underset{A_n}{\approx} \beta^{-1}$). Let $\beta\in C^\alpha$ of S_n where $\beta=\lambda_1\lambda_2...\lambda_{c(\beta)}$, λ_i are disjoint cycle factors and $|<\lambda_i>|=\alpha_i$, $(1\le i\le c(\beta)) \Rightarrow$ for each λ_i we have $\lambda_i=(b_1{}^i,b_2{}^i,...,b^i{}_{\alpha_i})$. So, $4\,|\,(\alpha_i-1) \Rightarrow \frac{(\alpha_i-1)}{4}=M\in Z$ (integer number) $\Rightarrow \frac{(\alpha_i-1)}{2}=2M \Rightarrow \frac{(\alpha_i-1)}{2}$ is even number. Let $\mu_i=(b_2^i,b_{\alpha_i}^i)(b_3^i,b_{\alpha_i-1}^i)(b_4^i,b_{\alpha_i-2}^i)....$, then we have $\mu_i\lambda_i\mu_i{}^{-1}=\lambda^{-1}{}_i$. Now we want to show that μ_i is an even permutation ($i.e$ $\mu_i\in A_n$) since μ_i is a composite of $\frac{(\alpha_i-1)}{2}$ (The number of transpositions is even) $\Rightarrow \mu_i\in A_n$. So for each $\lambda_i\ (1\le i\le c(\beta))$, $\exists\ \mu_i\in A_n$ such that $\mu_i\lambda_i\mu_i{}^{-1}=\lambda^{-1}{}_i$. Let $\mu=\mu_1\mu_2...\mu_{c(\beta)} \Rightarrow \mu\in A_n$ (from 2.6 (4)). Also, since λ_i $(1\le i\le c(\beta))$ disjoint cycles, then $\text{supp}(\lambda_i^{-1})\bigcap \text{supp}(\lambda^{-1}{}_j)=\phi$, for each $(1\le i\ne j\le c(\beta))$. If $\text{supp}(\mu_i)\bigcap\text{supp}(\mu_j)\ne\phi$ or $\text{supp}(\mu_i)\bigcap\text{supp}(\lambda_j)\ne\phi$ for some $(1\le i\ne j\le c(\beta)) \Rightarrow \exists\ b\in\{1,2,...,n\}$ such that $b\in\text{supp}(\lambda_i{}^{-1})\bigcap\text{supp}(\lambda^{-1}{}_j)$ for some $(1\le i\ne j\le c(\beta))$ (but this contradiction). Then $\text{supp}(\mu_i)\bigcap\text{supp}(\mu_j)=\phi$ and $\text{supp}(\mu_i)\bigcap\text{supp}(\lambda_j)=\phi$ $\forall(1\le i\ne j\le c(\beta)) \Rightarrow \mu_i\mu_j=\mu_j\mu_i$ and $\mu_i\lambda_j=\lambda_j\mu_i$ for each $(1\le i\ne j\le c(\beta))$. So

$$\mu\beta\mu^{-1}=\mu_1\mu_2...\mu_{c(\beta)}\lambda_1\lambda_2...\lambda_{c(\beta)}(\mu_1\mu_2...\mu_{c(\beta)})^{-1}$$
$$=\mu_1\mu_2...\mu_{c(\beta)}\lambda_1\lambda_2...\lambda_{c(\beta)}\mu_{c(\beta)}{}^{-1}...\mu_2{}^{-1}\mu_1{}^{-1}$$
$$=\mu_1\mu_2...\mu_{c(\beta)-1}\lambda_1\lambda_2...\mu_{c(\beta)}\lambda_{c(\beta)}\mu_{c(\beta)}{}^{-1}...\mu_2{}^{-1}\mu_1{}^{-1}$$
$$=\mu_1\mu_2...\mu_{c(\beta)-1}\lambda_1\lambda_2...\lambda_{c(\beta)}{}^{-1}...\mu_2{}^{-1}\mu_1{}^{-1}=$$
$$\lambda_1^{-1}\lambda_2^{-1}..\lambda_{c(\beta)}{}^{-1}=(\lambda_{c(\beta)}..\lambda_2\lambda_1)^{-1}=(\lambda_1\lambda_2..\lambda_{c(\beta)})^{-1}=\beta^{-1}.$$

Finally for each $\beta\in C^\alpha=C^{\alpha^+}\bigcup C^{\alpha^-} \Rightarrow \beta\in C^{\alpha^+}$ or $\beta\in C^{\alpha^-}$, there exists even permutation $\mu\in A_n$ such that $\mu\beta\mu^{-1}=\beta^{-1}$. So β is conjugate to its inverse in A_n. Then $\psi(\beta)=1$, for all $\beta\in H_n$ and hence G_ψ is a permutation connected graph.

Corollary 3.21:
Let $G_\psi=(H_n,E)$ be a permutation graph and $H_n=C^\alpha(\beta)$, where $n>1$ and $C^\alpha(\beta)$ is a conjugacy class of S_n. Then G_ψ is a permutation connected graph if

(i) The nonzero parts of $\alpha(\beta)$ are different and odd.
(ii) $4\,|\,(\alpha_i-1)$ for each parts α_i of $\alpha(\beta)$.

Proof: Since the nonzero parts of $\alpha(\beta)$ are different and odd, thus $C^\alpha(\beta)$ splits into two A_n-classes $C^{\alpha\pm}$ by [Theorem (2.3)]. Then $H_n=C^{\alpha+}\bigcup C^{\alpha-}$, but $4\,|\,(\alpha_i-1)$ for each parts α_i of α for the partition of β Hence G_ψ is a permutation connected graph by [Theorem (3.20)]

Example: 3.22: Let $G_\psi=(H_{15},E)$ be a permutation graph and $H_{15}=C^\alpha(\beta)$, where $\beta=(6)(1\ 3\ 2\ 9\ 15)(12\ 13\ 4\ 10\ 11\ 8\ 5\ 7\ 14)$

is a permutation in S_{15}. Then $\alpha(\beta)=(\alpha_1,\alpha_2,\alpha_3)=(1, 5, 9)$. Therefore $C^{\alpha}(\beta)$ splits into two A_{15}-classes $C^{\alpha\pm}$. So, $H_{15}=C^{\alpha+}\cup C^{\alpha-}$, but $4|(\alpha_i-1)$ for all $(i=1,2,3)$. Then $C^{\alpha\pm}$ are ambivalent in A_{15} and hence G_{ψ} is a permutation connected graph [since $\psi(\lambda)=1$, for all $\lambda\in H_{15}$]. Also, for anypermutation λ in H_{15} will be as the form $\lambda=(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9)(b_1,b_2,b_3,b_4,b_5)(c)\in H_{15}$. In other side, we consider that $\lambda^2=(a_1,a_3,a_5,a_7,a_9,a_2,a_4,a_6,a_8)(b_1,b_3,b_5,b_2,b_4)(c)\in H_{15}$ too. But H_{15} splits into two A_{15}-classes $C^{\alpha\pm}$ by [Theorem (2.3)] and $C^{\alpha\pm}$ are ambivalent by [Theorem (3.20)]. Then $\lambda^2 \underset{A_{15}}{\approx} (\lambda^2)^{-1}$ and this implies that $\psi(\lambda^2)=1$, for all $\lambda\in H_{15}$. However, $\psi(\lambda)=1$ for all $\lambda\in H_{15}$. That means $\psi(\lambda^2)=\psi(\lambda)$ for any $\lambda\in H_{15}$ and hence there are $|H_{15}|=\frac{n!}{z_\alpha}=\frac{(15)!}{45}=(29059430400)$ loops, where $\left|C^{\alpha}\right|=\frac{n!}{z_\alpha}$ with $z_\alpha=\prod_{r=1}^{n} r^{c_r}(c_r)!$ and $c_r=c_r^{(n)}(\beta)=\left|\{i:\alpha_i=r\}\right|$ (see Bump, 2004). Also, if $G_{\psi}=(H_5,E)$ is a permutation graph and $H_5=C^{\alpha}(\beta)$, where $\beta=(1\ 3\ 2\ 4\ 5)$ is a permutation in S_5. Then $\alpha(\beta)=(\alpha_1)=(5)$. Therefore $C^{\alpha}(\beta)$ splits into two A_5-classes $C^{\alpha\pm}$.

We can show each one of these classes $C^{\alpha\pm}$ with their permutations as following:

$C^{\alpha+}(\beta)=\{\beta_1=(1\ 2\ 3\ 5\ 4),\beta_2=(1\ 5\ 4\ 2\ 3),\beta_3=(1\ 2\ 5\ 4\ 3),\beta_4=(1\ 3\ 2\ 4\ 5),\beta_5=(1\ 4\ 5\ 3\ 2),\beta_6=(1\ 4\ 2\ 5\ 3),\beta_7=(1\ 3\ 5\ 2\ 4),\beta_8=(1\ 2\ 4\ 3\ 5),\beta_9=(1\ 4\ 3\ 2\ 5),\beta_{10}=(1\ 5\ 2\ 3\ 4),$
$\beta_{11}=(1\ 5\ 3\ 4\ 2),\ \beta_{12}=(1\ 3\ 4\ 5\ 2)\}$.
$C^{\alpha-}(\beta^{\#})=\{\beta_{13}=(1\ 2\ 3\ 4\ 5),\beta_{14}=(1\ 5\ 4\ 3\ 2),\beta_{15}=(1\ 2\ 5\ 3\ 4),\beta_{16}=(1\ 3\ 2\ 5\ 4),\beta_{17}=(1\ 4\ 5\ 2\ 3),$
$\beta_{18}=(1\ 4\ 3\ 5\ 2),$
$\beta_{19}=(15243),\beta_{20}=(15324),\beta_{21}=(13425),\beta_{22}=(14235),$
$\beta_{23}=(1\ 3\ 5\ 4\ 2),\beta_{24}=(1\ 2\ 4\ 5\ 3),\}$.

Where $\beta^{\#}=(1\ 2\ 5\ 3\ 4)$ and for any permutation $(a_1,a_2,a_3,a_4,a_5)\in C^{\alpha+}(\beta)$ we have $(a_1,a_3,a_5,a_2,a_4)\in C^{\alpha-}(\beta^{\#})$ (see [19]). In other words, $H_5=C^{\alpha+}\cup C^{\alpha-}$, but $4|(\alpha_1-1)$. Then $C^{\alpha\pm}$ are ambivalent in A_5 and hence G_{ψ} is a permutation connected graph [since $\psi(\lambda)=1$, for all $\lambda\in H_5$]. Also, for any permutation λ in H_5 will be as the form $\lambda=(a_1,a_2,a_3,a_4,a_5)\in H_5$. In other side, we consider that $\lambda^2=(a_1,a_3,a_5,a_2,a_4)\in H_5$ too. But H_5 splits into two A_5-classes $C^{\alpha\pm}$ by [Theorem (2.3)] and $C^{\alpha\pm}$ are ambivalent by [Theorem (3.20)]. Then $\lambda^2 \underset{A_5}{\approx} (\lambda^2)^{-1}$ and this implies that $\psi(\lambda^2)=1$, for all $\lambda\in H_5$. However, $\psi(\lambda)=1$ for all $\lambda\in H_5$. That means $\psi(\lambda^2)=\psi(\lambda)$ for any $\lambda\in H_5$ and hence there are $|H_5|=\frac{n!}{z_\alpha}=\frac{(5)!}{5}=(24)$ loops [see figure (3)].

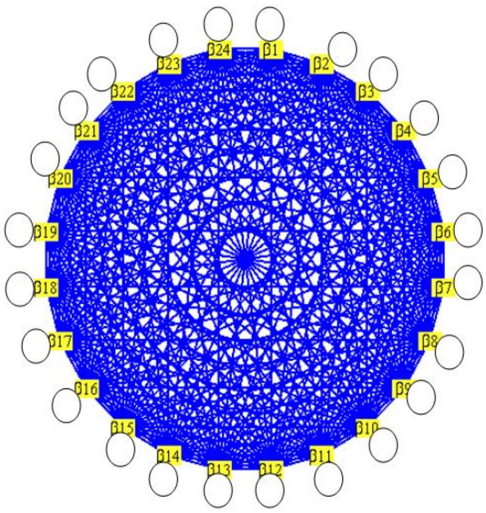

Figure 3:A graph has two connected permutation components.

Lemma 3.23:
Let $G_{\psi}=(H_n,E)$ be a permutationgraphand $H_n=C^{\alpha}(\beta)\cup C^{\alpha}(\lambda)$, where $n>1$, β,λ are even permutation in S_n. Then G_{ψ} is a permutation disconnected graph if

(i) The nonzero parts of $\alpha(\beta)$ are different and odd.

(ii) 4 does not divided (α_i-1) for some parts α_i of $\alpha(\beta)$.

(iii) $2|\alpha_i$ for some parts α_i of $\alpha(\lambda)$.

Proof: Since the nonzero parts of $\alpha(\beta)$ are different and odd, then $C^{\alpha}(\beta)$ splits into two A_n-classes $C^{\alpha\pm}$ by [Theorem (2.3)]. Then $H_n = C^{\alpha+} \cup C^{\alpha-} \cup C^{\alpha}(\lambda)$, but the number 4 does not divided $(\alpha_i - 1)$ for some parts α_i of $\alpha(\beta)$. Then $C^{\alpha+}$ and $C^{\alpha-}$ are non-ambivalent A_n-classes. That means β is even permutation exists in one of these classes and β^{-1} exists in the other, therefore $\psi(\beta)=1$. Also, since $2|\alpha_i$ for some parts α_i of $\alpha(\lambda)$. Then α_i is not odd for some parts α_i of $\alpha(\lambda)$.Hence $C^{\alpha}(\lambda)$ is not splits into two A_n-classes $C^{\alpha\pm}$. Then the cojugacy class $C^{\alpha}(\lambda)$ in S_n is the same in A_n. That means $\lambda \underset{A_n}{\approx} \lambda^{-1}$ (since $\lambda \underset{S_n}{\approx} \lambda^{-1}$, for any $\lambda \in S_n$).Hence $\psi(\lambda)=0$. Then $\psi(\beta) \neq \psi(\lambda)$ and hence G_{ψ} is a permutation disconnected graph.

Corollary 3.24: Let $G_{\psi} = (H_n, E)$ be a permutation graph and $H_n = C^{\alpha}(\beta) \cup C^{\alpha}(\lambda)$, where $n>1$, β, λ are even permutation in S_n. Then G_{ψ} has exactly two connected permutation components if
(i) The nonzero parts of $\alpha(\beta)$ are different and odd.
(ii) 4 does not divided $(\alpha_i - 1)$ for some parts α_i of $\alpha(\beta)$.
(iii) $2|\alpha_i$ for some parts α_i of $\alpha(\lambda)$.

Proof: By [Theorem (3.23)], we consider that G_{ψ} is a permutation disconnected graph and for any permutation $\lambda \in H_n = C^{\alpha}(\beta) \cup C^{\alpha}(\lambda)$ we have either $\lambda \in C^{\alpha}(\beta)$ or $\lambda \in C^{\alpha}(\lambda)$. This implies that either $\psi(\lambda)=1$ or $\psi(\lambda)=0$. Hence there are only two connected permutation components.

Example: 3.25: Let $G_{\psi} = (H_4, E)$ be a permutationgraphand $H_4 = C^{\alpha}(\beta) \cup C^{\alpha}(\gamma)$, where $\beta = (2)(1\ 3\ 4), \lambda = (2\ 3)(1\ 4)$ are even permutation in S_4. Then G_{ψ} has exactly two connected permutation components (Since $\forall \lambda \in H_n$, we have either $[\psi(\lambda)=0$, if $\lambda \in C^{\alpha}(\beta)]$ or $[\psi(\lambda)=1$, if $\lambda \in C^{\alpha}(\gamma)]$)where $C^{\alpha}(\beta)$={ β_1 =(4)(123), β_2 = (3)(124), β_3 = (4)(132), β_4 = (2)(134), β_5 = (3)(142), β_6 = (2)(143), β_7 = (1)(234), β_8 = (1)(243)} and $C^{\alpha}(\lambda)$={ λ_1 = (12)(34), λ_2 = (13)(24), λ_3 = (23) (14)}. That means the first subset $C^{\alpha}(\beta) = \{\beta_i \mid 1 \le i \le 8\}$ of H_4 contains all the vertices of the first connected permutation component and the second subset $C^{\alpha}(\gamma) = \{\lambda_1, \lambda_2, \lambda_3\}$ of H_4 contains all the vertices of the second connected permutation component. Also, $(\lambda_i, \lambda_j) \in E$ and $(\beta_i, \beta_j) \in E$, for any $\lambda_i, \lambda_j \in C^{\alpha}(\gamma)$ & $\beta_i, \beta_j \in C^{\alpha}(\beta)$. Therefore, $G_{\psi} = (H_4, E)$ has two permutation subgraphs. However, $(\lambda_j, \beta_i) \notin E$ and $(\beta_i, \lambda_j) \notin E$ for any $\lambda_j \in C^{\alpha}(\gamma)$ & $\beta_i \in C^{\alpha}(\beta)$ [Since $\psi(\lambda_j) \neq \psi(\beta_i)$]. That means there is no path between these two permutation subgraphs. Also, since $\psi(\lambda^2) = \begin{cases} 0, \ if\ \lambda \in C^{\alpha}(\beta) \\ 1, \ if\ \lambda \in C^{\alpha}(\gamma) \end{cases}, \forall \lambda \in H_n$, then there are $|H_4| = |C^{\alpha}(\beta)| + |C^{\alpha}(\gamma)| = 8+3 = 11$ loops in the permutation graph $G_{\psi} = (H_4, E)$.

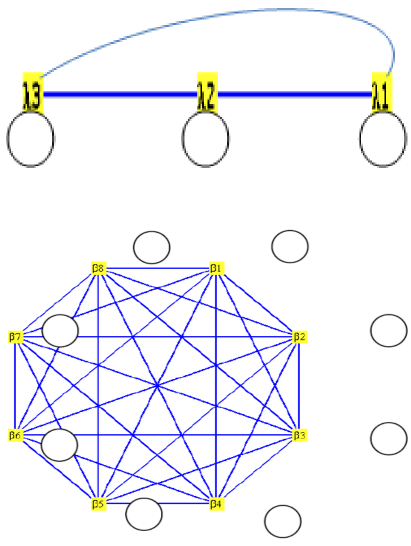

Figure 4:A graph has two connected permutation components.

CONCLUSION

For any permutation γ in symmetric group the new operations $\widehat{c}(\gamma)$ and $\psi(\gamma)$ have been introduced in this work to investigate new permutation graphs $G_{\widehat{c}}$

and G_{ψ}. Next, we study some of the essential properties of permutation graphs $G_{\bar{c}}$ and G_{ψ} with their permutations in as the vertices and discuss their structures as conjugacy classes in symmetric and alternating groups. In particular, we study the connected permutation graphs of graph theory. In future work, new classes of super-connected permutation graphs will be given and new equivalence relations will be found to partition the set $H_n \subseteq S_n$ into equivalence classes $C_1, C_2, \ldots, C_k$, under the relation that vertices γ_1 and γ_2 are equivalent iff there is a path from γ_1 to γ_2 and another from γ_2 to γ_1.

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