QUOTIENT MV-ALGEBRAS

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Abstract— Let X be an MV-algebra. We establish the quotient MV-algebra of X modulo an idea of X which is again an MV-algebra. Next we study homomorphism on the quotient MV-algebra and investigate related properties.

Index Terms—Ideal, congruence, quotient MV-algebra.

I. INTRODUCTION

C. C. Chang [2] introduced the notion of an MV-algebra and developed a theory of the MV-algebra to provide an algebraic proof of the completeness theorem of infinite valued Lukasiewicz propositional calculus [3]. A. Filipoiu, G. Georgescu and A. Lettieri [5] defined a maximal MV-algebra. A concept of a maximal MV-algebra is similar to the maximal rings and maximal distributive lattices. They proved that a maximal MV-algebra is semilocal [4] and characterized a maximal MV-algebra as finite direct product of local maximal MV-algebras [1]. A. D. Nola, R. Grigolia and G. Panti [6] characterized the automorphism groups of a free MV-algebra over finitely many generators. S. Rasouli and B. Davvaz [7] introduced the notion of rough ideal with respect to an ideal of an MV-algebra and gave some properties of the lower and the upper approximations in an MV-algebra. In this paper, we study the quotient MV-algebra and investigate its homomorphism properties.

II. PRELIMINARIES

Definition1. An MV-algebra is a structure $(X,\oplus,*,0)$ with a binary operation $\oplus$, a unary operation $*$, and a distinguished constant 0 satisfying the following axioms: for any $x,y,z\in X$:

1) $(X,\oplus,0)$ is a commutative monoid,
2) $(x^*)^*=x$,
3) $0^*\oplus x=0^*$,
4) $(x^*\oplus y)^*\oplus y=(y^*\oplus y)^*\oplus x$.

A subalgebra of an MV-algebra X is a subset S of X containing the zero element of X, closed under the operations of X and equipped with the restriction to S of these operations. We define $1:=0^*$ and the auxiliary operations $\bullet$, $\wedge$ and $\vee$ by

$$x\bullet y:=(x^*\oplus y^*)^*, \qquad (1)$$

$$x\wedge y:=x\bullet(y\oplus x^*),$$

$$x\vee y:=x\oplus(y\bullet x^*).$$

Then $(X,\bullet,0)$ is a commutative monoid. We define the binary operation $\otimes$ by

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$$x\otimes y:=x\bullet y^*, \qquad (2)$$

and define a relation $\le$ on X by $x\le y$ if and only if $x\wedge y=x$ for each $x,y\in X$. Then $(X,\le)$ is a partially ordered set. The distance function in X is defined by

$$d:X\times X\to X,\ d(x,y):=(x^*\bullet y)\oplus(y^*\bullet x).$$

Proposition2. [2] Let X be an MV-algebras. The following properties hold for all $x,y,z,u,v\in X$:

1) $d(x,y)=d(y,x)$,
2) $d(x,y)=0$ if and only if $x=y$,
3) $d(x,0)=x$ and $d(x,1)=x^*$,
4) $d(x^*,y^*)=d(x,y)$,
5) $d(x,y)\le d(x,z)\oplus d(z,x)$,
6) $d(x\oplus u,y\oplus v)\le d(x,y)\oplus d(u,v)$,
7) $x\otimes x=0$,
8) $(x\otimes y)\oplus y=(y\otimes x)\oplus x$.

Definition3. Let X be an MV-algebra and A be a non-empty subset of X. Then A is called an ideal of X if the following conditions are satisfied:

1) $0\in A$,
2) $x,y\in A$ implies $x\oplus y\in A$,
3) $x\in A$ and $y\le x$ imply $y\in A$.

Note that the intersection of two different ideals of an MV-algebra X is again an ideal of X, but the union is not necessary. If A is a non-empty subset of X. Then the set

$$<A>=\{x\in X\mid x\le a_1\oplus\cdots\oplus a_n,\forall a_i\in A,n\in \mathrm{N}\}$$

is the smallest ideal of X containing A. A proper ideal P of X is said to be prime if and only if whenever $x\wedge y\in P$ then either $x\in P$ or $y\in P$. A proper ideal M of X is said to be maximal if and only if for any ideal I of X such that $M\subset I$, either $M=I$ or $I=X$. The set of all prime ideals of X is denoted by $Spec(X)$ and the set of maximal ideals of X is denoted by $Max(X)$. One can show that a maximal ideal of X is prime, i.e. $Spec(X)\subset Max(X)$.

III. QUOTIENT MV-ALGEBRA

Let A be an ideal of an MV-algebra X. Now we define $x \equiv_A y$ if and only if $d(x,y) \in X$.

Then the relation $\equiv_A$ is a congruence relation on X, i.e. $\equiv_A$ is reflexive, symmetric, transitive, and satisfies the following: $x \equiv_A y$ and $u \equiv_A v$ imply $x \oplus u \equiv_A y \oplus v$ and $x^* \equiv_A y^*$. The equivalence class of an element x of X is denoted as the set

$$[x]_A = \{y \in X \mid x \equiv_A y\}.$$

Then the following always hold for all $x, y \in X$: $x \in [x]_A$,

$$x \equiv_A y \text{ if and only if } [x]_A = [y]_A,$$

and every two equivalence classes are either equal or disjoint. We shall leave the proof as an exercise.

Let A be an ideal of an MV-algebra $(X, \oplus, *, 0)$. We define the quotient set

$$X/A = \{[x]_A \mid x \in X\},$$

and define a binary operation $\oplus$ and a unary operation * on X/A as follows:

$$[x]_A \oplus [y]_A = [x \oplus y]_A \text{ and } [x]_A^* = [x^*]_A.$$

For an ideal A of an MV-algebra X, the quotient set X/A is called the quotient MV-algebra. The following theorem shows that a quotient MV-algebra is an MV-algebra.

Theorem4. Let A be an ideal of an MV-algebra $(X, \oplus, *, 0)$. Then the quotient set $(X/A, \oplus, *, [0]_A)$. is an MV-algebra.

Proof. First we have to show that the operation $\oplus$ is well-defined on X/A. Suppose that $[x]_A = [y]_A$ and $[a]_A = [b]_A$. Then we have that $x \equiv_A y$ and $a \equiv_A b$. Since $\equiv_A$ is a congruence relation, it follows that

$$(x \oplus a) \equiv_A (y \oplus b).$$

So we have

$$\begin{aligned} [x]_A \oplus [a]_A &= [x \oplus a]_A \\ &= [y \oplus b]_A \\ &= [y]_A \oplus [b]_A. \end{aligned}$$

Hence $\oplus$ is well-defined.

Now let $x, y \in X$. Then $([x]_A^*)^* = ([(x^*)^*]_A = [x]_A$,

$$\begin{aligned} ([x]_A^* \oplus [y]_A)^* \oplus [y]_A &= [(x^* \oplus y)^* \oplus y]_A \\ &= [(y^* \oplus y)^* \oplus x]_A \\ &= ([y]_A^* \oplus [y]_A)^* \oplus [x]_A \end{aligned}$$

and $[0]_A^* \oplus [x]_A = [0^* \oplus x]_A = [0^*]_A = [0]_A^*$. One can easily check that $(X/A, \oplus, [0]_A)$ is a commutative monoid. Therefore X/A is an MV-algebra.

Proposition5. In a quotient MV-algebra X/A, the property $[x]_A \otimes [y]_A = [x \otimes y]_A$ always holds for all x and y in X.

Proof. Let $x, y \in X$. Using (1) and (2) we obtain

$$\begin{aligned} [x]_A \otimes [y]_A &= [x]_A \bullet [y]_A^* \\ &= ([x]_A^* \oplus [y]_A)^* \\ &= [(x^* \oplus y)^*]_A \\ &= [x \bullet y^*]_A \\ &= [x \otimes y]_A. \end{aligned}$$

Let X and Y be MV-algebras. Let f be a map from X into Y. f is said to be homomorphism if and only if the following conditions are satisfied for all $x, y \in X$: $f(0) = 0$, $f(x \oplus y) = f(x) \oplus f(y)$ and $f(x^*) = f(x)^*$. The kernel of a homomorphism f is the set

$$\ker f = \{x \in X \mid f(x) = 0\}.$$

One can show that $\ker f$ is an ideal of X.

Proposition6. Let f be a homomorphism from an MV-algebra X into an MV-algebra Y. Then for all $x, y \in X$, $f(x \otimes y) = f(x) \otimes f(y)$.

Proof. Let $x, y \in X$. Then

$$\begin{aligned} f(x \otimes y) &= f(x \bullet y^*) \\ &= f((x^* \oplus y)^*) \\ &= f(x^* \oplus y)^* \\ &= (f(x)^* \oplus f(y))^* \\ &= f(x) \bullet f(y)^* \\ &= f(x) \otimes f(y). \end{aligned}$$

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